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Affine Toda field theories with defects

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Abstract: A lagrangian approach is proposed and developed to study defects within affine Toda field theories. In particular, a suitable Lax pair is constructed together with examples of conserved charges. It is found that only those models based on $a_r^{(1)}$ data appear to allow defects preserving integrability. Surprisingly, despite the explicit breaking of Lorentz and translation invariance, modified forms of both energy and momentum are conserved. Some, but apparently not all, of the higher spin conserved charges are also preserved after the addition of contributions from the defect. This fact is illustrated by noting how defects may preserve a modified form of just one of the spin 2 or spin -2 charges but not both of them.

Keywords: Field Theories in Lower Dimensions, Integrable Field Theories.

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1. Introduction

In a recent article, based principally on the examples of the sinh-Gordon, Liouville, or free scalar field models [1], it was pointed out that field theories in $1+1$ dimensions may have internal boundary conditions with interesting consequences. Typically, at an internal boundary the classical field will have a discontinuity, hence the name 'defect', yet energy and momentum will be conserved once they have been suitably modified. In addition, any number of defects may be placed along the x-axis, and each will introduce additional parameters to the model. The purpose of this article is to explore the consequences of integrability in the presence of defects.

The quantum version of a situation similar to this has been examined before and imposing the requirements of integrability was found to be highly restrictive. The investigation was pioneered by Delfino, Mussardo and Simonetti some years ago [2] and there has been renewed interest in it recently [3, 4].

Over some years, there has been interest in the study of integrable classical or quantum field theories restricted to a half-line, or interval, by imposing integrable boundary conditions, say at $x=0$, see for example [5]–[11]. The simplest situation, which is also the best understood, contains a real self-interacting scalar field ϕ with either a periodic (cos), or non-periodic (cosh) potential. The sinh-Gordon model can be restricted to the left half-line $-\infty \le x \le 0$, without losing integrability, by imposing the boundary condition

$$\partial_x \phi|_{x=0} = \frac{\sqrt{2}m}{\beta}\left(\varepsilon_0 e^{-\frac{\beta}{\sqrt{2}}\phi(0,t)} - \varepsilon_1 e^{\frac{\beta}{\sqrt{2}}\phi(0,t)}\right), \tag{1.1}$$

where m and β are the bulk mass scale and coupling constant, respectively, and ε_0 and ε_1 are two additional parameters [6, 10]. This set of boundary conditions generally breaks the reflection symmetry $\phi \to -\phi$ of the model although the symmetry is explicitly preserved when $\varepsilon_0 = \varepsilon_1 \equiv \varepsilon$. The restriction of the sinh-Gordon model to a half-line is a considerable complication, and renders the model more interesting than it appears to be in the bulk. This is because generally there will be additional states in the spectrum associated with the boundary, together with a set of reflection factors which must be compatible with the bulk S-matrix (see [6, 7, 9, 12]). The weak-strong coupling duality enjoyed by the bulk theory emerges in a new light [13, 14, 15].

Integrability in the bulk sinh-Gordon model requires the existence of conserved quantities labelled by odd spins $s = \pm 1, \pm 3, \ldots$, and some of these should survive even in the presence of boundary conditions. Since boundary conditions typically violate translation invariance, it would be expected that the 'momentum-like' combinations of conserved quantities should not be preserved. However, the 'energy-like' combinations, or some subset of them, might remain conserved, at least when suitably modified (see [6] for the paradigm). As is the case for the theory restricted to a half-line, it was reported in [1] that the spin three charge already supplies the most general restrictions on the internal boundary condition in the presence of a defect. The Lax pair approach developed in [11], adapted to the new context, was used to re-derive the boundary conditions, thereby demonstrating that the preservation of higher spin energy-like charges imposed no further restrictions on the internal boundary conditions. More surprisingly, it was noted that after suitable modification momentum was conserved.

In this article, it is intended to give a fuller discussion of defects from a lagrangian point of view, and to examine the possibilities within the context of affine Toda field theories. The basic ideas will be illustrated by a free massive real scalar field since for that case there are no complications arising from the requirements of integrability. Later, the Lax pair will be used to establish the existence of defects within a certain class of affine Toda field theory - those based on the root data of $a_r^{(1)}$ - the simplest example of these being the already-discussed sinh-Gordon model, based on $a_1^{(1)}$ data.

2. Real scalar field

For convenience, the field in the region $x > 0$ will be denoted ψ and the field in the region $x < 0$ will be denoted ϕ; in principle, each could have its own bulk potential denoted $W(\psi)$ and $V(\phi)$, respectively. The portion of the lagrangian associated with the defect is taken to be

$$\mathcal{L}_D = \delta(x)\left[\frac{1}{2}\left(\phi\partial_t\psi - \psi\partial_t\phi\right) - \mathcal{B}(\phi,\psi)\right], \tag{2.1}$$

where $\mathcal{B}(\phi,\psi)$ is the defect potential, and therefore the defect conditions at $x = 0$ are:

$$\begin{aligned}\partial_x\phi &= \partial_t\psi - \frac{\partial\mathcal{B}}{\partial\phi}\\ \partial_x\psi &= \partial_t\phi + \frac{\partial\mathcal{B}}{\partial\psi}\,.\end{aligned} \tag{2.2}$$

Consider the time derivative of the standard bulk momentum, which is not expected to be preserved because of the lack of translation invariance. Then, ignoring contributions from $\pm\infty$, this may be written in terms of the fields and their derivatives evaluated at the boundary $x = 0$:

$$\begin{aligned} \frac{dP}{dt} &= \frac{d}{dt}\left(\int_{-\infty}^{0} dx\,(\partial_t\phi\partial_x\phi) + \int_{0}^{\infty} dx\,(\partial_t\psi\partial_x\psi)\right) \\ &= \frac{1}{2}\left[(\partial_x\phi)^2 + (\partial_t\phi)^2 - (\partial_x\psi)^2 - (\partial_t\psi)^2 - 2V(\phi) + 2W(\psi)\right]_{x=0}. \end{aligned} \tag{2.3}$$

Using the boundary conditions (2.2), the combination of terms evaluated at the boundary may be rewritten

$$-\partial_t\psi\frac{\partial\mathcal{B}}{\partial\phi} - \partial_t\phi\frac{\partial\mathcal{B}}{\partial\psi} - V(\phi) + W(\psi) + \frac{1}{2}\left(\frac{\partial\mathcal{B}}{\partial\phi}\right)^2 - \frac{1}{2}\left(\frac{\partial\mathcal{B}}{\partial\psi}\right)^2,$$

which in turn is a total time-derivative of a functional of $\phi(0,t)$ and $\psi(0,t)$, provided

$$\frac{\partial^2\mathcal{B}}{\partial\phi^2} - \frac{\partial^2\mathcal{B}}{\partial\psi^2} = 0\,, \tag{2.4}$$

$$\frac{1}{2}\left(\frac{\partial\mathcal{B}}{\partial\phi}\right)^2 - \frac{1}{2}\left(\frac{\partial\mathcal{B}}{\partial\psi}\right)^2 = V(\phi) - W(\psi)\,. \tag{2.5}$$

There are many solution to equations (2.4) but, for free fields, the simplest would be to take

$$V(\phi) = \frac{1}{2}m^2\phi^2\,, \qquad W(\psi) = \frac{1}{2}m^2\psi^2 \tag{2.6}$$

$$\mathcal{B} = \left[\frac{m\lambda}{4}(\phi+\psi)^2 + \frac{m}{4\lambda}(\phi-\psi)^2\right]_{x=0}, \tag{2.7}$$

where λ is a free parameter. Clearly, there is no requirement for ϕ and ψ to match at $x = 0$, except in the limit $\lambda \to 0$. With the expressions (2.6) the combination

$$\mathcal{P} = P + \left[\frac{m\lambda}{4}(\phi+\psi)^2 - \frac{m}{4\lambda}(\phi-\psi)^2\right]_{x=0} \tag{2.8}$$

is readily checked to be a conserved quantity, as is the total energy

$$\mathcal{E} = E + \left[\frac{m\lambda}{4}(\phi+\psi)^2 + \frac{m}{4\lambda}(\phi-\psi)^2\right]_{x=0}. \tag{2.9}$$

Indeed, in light-cone coordinates:

$$\mathcal{P}_{\pm 1} = \mathcal{E} \pm \mathcal{P} = E \pm P + \left[\frac{m\lambda^{\pm 1}}{2}(\phi\pm\psi)^2\right]_{x=0}, \tag{2.10}$$

indicating that the parameter λ introduced at the defect, although breaking Lorentz invariance, has naturally the character of a spin one quantity.

For the linear system, the defect simply causes a delay. Thus with,

$$\phi = e^{-i\omega t + ikx} + Re^{-i\omega t - ikx}\,, \quad x < 0\,; \qquad \psi = Te^{-i\omega t + ikx}\,, \quad x > 0\,,$$

the boundary conditions require

$$R = 0\,, \qquad T = -\frac{ik + \frac{m}{2}\left(\lambda + \frac{1}{\lambda}\right)}{i\omega + \frac{m}{2}\left(\lambda - \frac{1}{\lambda}\right)}\,.$$

Clearly, neither the fields nor their spatial derivatives are continuous through the defect.

It is interesting to note (see [1]) that if the conditions (2.2) were regarded as a pair of differential equations in the bulk then the additional conditions on the boundary potential $\mathcal{B}$, summarised in (2.4), would guarantee the pair to be a Bäcklund transformation leading to

$$\partial^2\phi = -\frac{\partial V}{\partial\phi}\,, \qquad \partial^2\psi = -\frac{\partial W}{\partial\psi}\,. \tag{2.11}$$

Following from this remark, it is to be expected that any pair of fields ϕ, ψ related by a Bäcklund transformation [19] should be able to support a defect. In [1] it was pointed out that besides free massive fields this is indeed the case for sinh-Gordon, Liouville and massless free fields, assembled in suitable combinations. The close association of the Bäcklund transformation with a defect offers new insight into the Bäcklund transformation itself.

In this article the analysis is extended to other affine Toda field theories.

3. Lax pair for affine Toda field theory with a defect

The starting point for this discussion will be the standard affine Toda field theory lagrangian (see [16]) together with a defect contribution, which in the first instance is located at $x = 0$. The notation used above for the fields in the two regions $x < 0$ and $x > 0$ will be maintained although in the case of affine Toda field theory the fields will be multi-component. Bearing this in mind, a fairly general lagrangian, including a defect contribution, would be

$$\mathcal{L} = \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi + \delta(x)\left(\frac{1}{2}\phi E\partial_t\phi + \phi D\partial_t\psi + \frac{1}{2}\psi F\partial_t\psi - \mathcal{B}(\phi,\psi)\right), \tag{3.1}$$

where D, E and F are matrices independent of ϕ and ψ. Omitting total time derivatives requires E and F to be antisymmetric. More general possibilities might allow these to be functions of the fields, and might also allow for other than linear terms in time derivatives. For models on a half-line, the latter possibility has been considered by Baseilhac and Delius [17].

Following from the lagrangian (3.1), the defect conditions at $x = 0$ are:

$$\begin{aligned} \partial_x\phi - E\ \ \partial_t\phi - D\,\partial_t\psi + \nabla_\phi\mathcal{B} &= 0 \\ \partial_x\psi - D^T\partial_t\phi + F\,\partial_t\psi - \nabla_\psi\mathcal{B} &= 0\,. \end{aligned} \tag{3.2}$$

Using the ideas presented in [11] it is straightforward to set up a Lax pair which will automatically incorporate the boundary conditions (3.2). In the bulk, the Lax pair for affine Toda field theory is well-known and has the following form:

$$\begin{aligned} a_t &= \frac{1}{2}\left[\partial_x\phi\cdot\mathbf{H} + \sum_i \sqrt{m_i}e^{\alpha_i\cdot\phi/2}\left(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i}\right)\right] \\ a_x &= \frac{1}{2}\left[\partial_t\phi\cdot\mathbf{H} + \sum_i \sqrt{m_i}e^{\alpha_i\cdot\phi/2}\left(\lambda E_{\alpha_i} + \frac{1}{\lambda}E_{-\alpha_i}\right)\right], \end{aligned} \tag{3.3}$$

where $\mathbf{H}$ are the generators in the Cartan subalgebra of the semi-simple Lie algebra whose simple roots are α_i, $i = 1, \ldots, r$, and $E_{\pm\alpha_i}$ are the generators corresponding to the simple roots or their negatives. The additional root, given by $\alpha_0 = -\sum_i n_i \alpha_i$, is the (euclidean part of) the additional root in the Dynkin-Kač diagram and is appended to the set of simple roots. The two expressions (3.3) are readily checked to be a Lax pair using additional facts about the Lie algebra commutation relations (for more details about this, and further references, see [21]):

$$[\mathbf{H}, E_{\pm\alpha_i}] = \pm\alpha_i E_{\pm\alpha_i}\,, \qquad [E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij}\frac{2\alpha_i}{\alpha_i^2}\cdot\mathbf{H}\,, \tag{3.4}$$

and $m_i = n_i\alpha_i^2/2$. That is

$$\partial_t a_x - \partial_x a_t + [a_t, a_x] = 0 \quad \Leftrightarrow \quad \partial^2\phi = -\nabla_\phi V(\phi)\,, \qquad V(\phi) = \sum_{i=0}^{r} n_i e^{\alpha_i\cdot\phi}\,. \tag{3.5}$$

For ease of notation, the coupling β and the bulk mass parameter m are omitted but may be reinstated by appropriate rescalings. As usual, there is a family of Lax pairs labelled by the spectral parameter λ.

To incorporate a defect at $x = 0$, the Lax pair may be adapted as follows [11]. Extend the region $x < 0$ to $x < b$, $b > 0$ and the region $x > 0$ to $x > a$, $a < 0$ (in other words, consider two overlapping regions $R^<$ and $R^>$ each containing the defect), and in each region define a new Lax pair:

$$\begin{aligned}
R^<: \quad \widehat{a}_t^< &= a_t(\phi) - \frac{1}{2}\theta(x-a)\left(\partial_x\phi - E\;\partial_t\phi - D\,\partial_t\psi + \nabla_\phi\mathcal{B}\right)\cdot\mathbf{H}\\
\widehat{a}_x^< &= \theta(a-x)a_x(\phi)\\
R^>: \quad \widehat{a}_t^> &= a_t(\psi) - \frac{1}{2}\theta(b-x)\left(\partial_x\psi - D^T\partial_t\phi + F\,\partial_t\psi - \nabla_\psi\mathcal{B}\right)\cdot\mathbf{H}\\
\widehat{a}_x^> &= \theta(x-b)a_x(\psi)\,.
\end{aligned} \tag{3.6}$$

Checking directly reveals that eqs(3.6) yield both the equations of motion and defect conditions (displaced to $x = a$ and $x = b$, respectively) for the fields in the two regions $x < a$ and $x > b$. In the overlap region $a < x < b$, the components $\widehat{a}_x^>$, $\widehat{a}_x^<$ vanish, implying the other components, $\widehat{a}_t^>$, $\widehat{a}_t^<$, must be x-independent, in turn implying that both ϕ and ψ are independent of x throughout the overlap. On the other hand, maintaining the zero-curvature condition within the overlap also requires the two components $\widehat{a}_t^>$ and $\widehat{a}_t^<$ to be related by a gauge transformation:

$$\partial_t\mathcal{K} = \mathcal{K}a_t^> - a_t^<\mathcal{K}\,, \tag{3.7}$$

where $\mathcal{K}$ is a matrix of dimension equal to the dimension of the representation chosen for the Lie algebra generators $\mathbf{H}$, $E_{\pm\alpha_i}$. With the given boundary conditions, $\mathcal{K}$ will be t-dependent so it is convenient to make the following change:

$$\mathcal{K} = e^{-\frac{1}{2}\mathbf{H}\cdot(E\phi + D\psi)}\,\tilde{\mathcal{K}}\,e^{\frac{1}{2}\mathbf{H}\cdot(D^T\phi - F\psi)}\,. \tag{3.8}$$

Assuming $\tilde{\mathcal{K}}$ is t-independent and using the explicit expressions for the Lax pair, eq(3.7) provides a set of equations constraining the defect potential $\mathcal{B}$ together with the group element $\tilde{\mathcal{K}}$:

$$\tilde{\mathcal{K}}\mathbf{H}\cdot\nabla_\psi\mathcal{B}+\mathbf{H}\tilde{\mathcal{K}}\cdot\nabla_\phi\mathcal{B}=\sum_{i=0}^{r}\sqrt{m_i}\times \tag{3.9}$$
$$\times\left[\lambda\left(E_{\alpha_i}\tilde{\mathcal{K}}e^{\frac{1}{2}\alpha_i\cdot(\phi+E\phi+D\psi)}-\mathcal{K}E_{\alpha_i}e^{\frac{1}{2}\alpha_i\cdot(\psi-F\psi+D^T\phi)}\right)-\right.$$
$$\left.-\frac{1}{\lambda}\left(E_{-\alpha_i}\tilde{\mathcal{K}}e^{\frac{1}{2}\alpha_i\cdot(\phi-E\phi-D\psi)}-\tilde{\mathcal{K}}E_{-\alpha_i}e^{\frac{1}{2}\alpha_i\cdot(\psi+F\psi-D^T\phi)}\right)\right].$$

Bearing in mind the defect potential $\mathcal{B}$ does not depend on the spectral parameter, and that $\tilde{\mathcal{K}}$ is not expected to depend on the fields in either region (since it could not be t-independent otherwise), it turns out eq(3.9) is extremely strong and severely limits the possible defects. Note, $\tilde{\mathcal{K}}$ has a certain obvious arbitrariness since it may be multiplied by any function of λ, commuting with all generators of the Lie algebra, without affecting (3.9).

For example, if it is supposed that $\tilde{\mathcal{K}}$ has a finite limit as $\lambda\to\infty$, with $\tilde{\mathcal{K}}(\infty)=1$, then the terms of order λ on the right hand side of (3.9) must cancel. For that to be the case, given the fields and root data are real, the exponential functions appearing as coefficients of the Lie algebra generators must match exactly for each i. In other words,

$$\alpha_i\cdot(1+E-D^T)\phi=\alpha_i\cdot(1-F-D)\psi\,,\qquad i=0,\dots,r\,. \tag{3.10}$$

If it is further supposed that the defect is 'maximal', in the sense that the magnitude of the defect is not prescribed and there is no relation between the field values $\phi(t,0)$ and $\psi(t,0)$, then since the simple roots are linearly independent the three matrices D,E and F must satisfy

$$E=F=1-D\,,\qquad D+D^T=2\,. \tag{3.11}$$

Incorporating these relations into (3.9) gives

$$\tilde{\mathcal{K}}\mathbf{H}\cdot\nabla_\psi\mathcal{B}+\mathbf{H}\tilde{\mathcal{K}}\cdot\nabla_\phi\mathcal{B}=$$
$$=\sum_{i=0}^{r}\sqrt{m_i}\left[\lambda\left(E_{\alpha_i}\tilde{\mathcal{K}}e^{\frac{1}{2}\alpha_i\cdot(D^T\phi+D\psi)}-\tilde{\mathcal{K}}E_{\alpha_i}e^{\frac{1}{2}\alpha_i\cdot(D\psi+D^T\phi)}\right)-\right.$$
$$\left.-\frac{1}{\lambda}\left(E_{-\alpha_i}\tilde{\mathcal{K}}e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)}-\tilde{\mathcal{K}}E_{-\alpha_i}e^{\frac{1}{2}\alpha_i\cdot D^T(\psi-\phi)}\right)\right]. \tag{3.12}$$

An alternative possibility might allow $\tilde{\mathcal{K}}(\infty)$ to permute the simple roots,

$$\tilde{\mathcal{K}}(\infty)E_{\alpha_i}=E_{\alpha_{\pi(i)}}\tilde{\mathcal{K}}(\infty)\,.$$

Permutations of this kind are automorphisms of the Kač-Dynkin diagram for the affine roots. They will not be considered further in this article.

The next step makes the reasonable assumption

$$\tilde{\mathcal{K}} = \tilde{\mathcal{K}}(\infty) + \frac{k_1}{\lambda} + \frac{k_2}{\lambda^2} + \cdots \equiv 1 + \frac{k_1}{\lambda} + \frac{k_2}{\lambda^2} + \cdots\,, \tag{3.13}$$

and examining the terms of order λ^0 in (3.12) reveals

$$\mathbf{H}\cdot(\nabla_\phi\mathcal{B} + \nabla_\psi\mathcal{B}) = -\sum_{i=0}^{r}\sqrt{m_i}\,[k_1, E_{\alpha_i}]\,e^{\frac{1}{2}\alpha_i\cdot(D^T\phi + D\psi)}\,.$$

Hence,

$$k_1 = \sum_{i=0}^{r} c_i E_{-\alpha_i}\,, \qquad \mathcal{B} = \sum_{i=0}^{r} d_i\,e^{\frac{1}{2}\alpha_i\cdot(D^T\phi + D\psi)} + \tilde{\mathcal{B}}(\phi - \psi)\,, \qquad c_i = \frac{d_i\alpha_i^2}{2\sqrt{m_i}}\,, \tag{3.14}$$

where the defect potential is determined only partially at this stage, up to a term which is a function of the defect $\phi - \psi$ (both fields evaluated at $x = 0$), and a set of constants $d_i,\ i = 0, \ldots, r$.

The order $1/\lambda$ terms in (3.12) are more tricky to analyse. They give

$$k_1\mathbf{H}\cdot\nabla_\psi\mathcal{B} + \mathbf{H}k_1\cdot\nabla_\phi\mathcal{B} = -\sum_{i=0}^{r}\sqrt{m_i}\Bigg([k_2, E_{\alpha_i}]\,e^{\frac{1}{2}\alpha_i\cdot(D^T\phi + D\psi)} + E_{-\alpha_i}\left[e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)} - e^{\frac{1}{2}\alpha_i\cdot D^T(\psi-\phi)}\right]\Bigg)\,, \tag{3.15}$$

and it is rather clear this splits into two parts, one being an equation for k_2, effectively, and the other involving $\tilde{\mathcal{B}}$, the not yet determined part of the defect potential. The matrix D is not yet determined either, and therefore it is not yet clear to what extent the exponentials appearing in the second group of terms in (3.15) are independent. However, a sensible first guess for $\tilde{\mathcal{B}}$ is likely to be of the form

$$\tilde{\mathcal{B}} = \sum_{i=0}^{r}\left(p_i\,e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)} + q_i\,e^{\frac{1}{2}\alpha_i\cdot D^T(\psi-\phi)}\right)\,, \tag{3.16}$$

on the grounds that $\tilde{\mathcal{B}}$ depends only on the defect $\phi - \psi$. Taking this to be the case gives

$$\sum_{j=0}^{r}\alpha_j\cdot\left(p_j D\alpha_i\,e^{\frac{1}{2}\alpha_j\cdot D(\phi-\psi)} - q_j D^T\alpha_i\,e^{\frac{1}{2}\alpha_j\cdot D^T(\psi-\phi)}\right) = \frac{2\sqrt{m_i}}{c_i}\left(e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)} - e^{\frac{1}{2}\alpha_i\cdot D^T(\psi-\phi)}\right)\,. \tag{3.17}$$

For the simplest choice which uses the root data for $a_1^{(1)}$, $\alpha_1 = -\alpha_0 = \alpha$ and $D = 1$, the parameters in (3.16) appear in the combinations $p_0 + q_1$ and $p_1 + q_0$. To satisfy (3.17), $c_0 = c_1 = c$ and

$$p_0 + q_1 = \frac{2\sqrt{m}}{c\alpha^2} = p_1 + q_0\,, \qquad m_0 = m_1 = m\,. \tag{3.18}$$

Also, from (3.14) it is clear $d_0 = d_1 = d = 2c\sqrt{m}/\alpha^2$. That is, with the usual conventions, $p_0 + q_1 = 1/c$, $d_0 = d_1 = c$. Hence, there is a single free parameter c and the defect potential reported in [1] is recovered. For the case of $a_1^{(1)}$, there are no further constraints on the parameter c from those relations arising as coefficients of higher orders in λ. However, with other choices, the analysis is less straightforward.

First of all, it is clear $D = 1$ cannot work since the simple roots would have to be mutually orthogonal, which is never the case. An alternative would have to allow cancellation between different terms on the left hand side of (3.17). If there is a defect and $D + D^T = 2$ then cancellation between the two pieces for a specific j-value within the sum cannot happen. For cancellations to be possible between terms with different j-values it would be necessary to have

$$\alpha_j D = -\alpha_{\Pi^{-1}(j)} D^T \equiv -\alpha_j \pi^T D^T , \tag{3.19}$$

where Π permutes the simple roots, and π achieves the equivalent via an orthogonal transformation of α_j. Since the simple roots are a linearly independent set, eq(3.19) requires

$$D = -\pi^T D^T = 2 - D^T . \tag{3.20}$$

Hence,

$$D^T D = D D^T , \qquad D = 2(1-\pi)^{-1} , \qquad D^T = 2(1-\pi^T)^{-1} , \tag{3.21}$$

provided $1-\pi$ and $1-\pi^T$ have inverses. These inverses will fail to exist if π leaves a real linear combination of the simple roots invariant (in other words, π has an eigenvector with eigenvalue 1). Noting that the two sets of terms in the putative expression for $\tilde{\mathcal{B}}$ (3.16) are the same but ordered differently, it is clear that half the coefficients are redundant (as was found in the $a_1^{(1)}$ case described above). Thus a more refined expression is

$$\tilde{\mathcal{B}} = \sum_{i=0}^{r} s_i \, e^{\frac{1}{2}\alpha_i \cdot D(\phi-\psi)} , \tag{3.22}$$

and (3.17) will be replaced by

$$\sum_{j=0}^{r} s_j \, \alpha_j \cdot D\alpha_i \, e^{\frac{1}{2}\alpha_j \cdot D(\phi-\psi)} = \frac{2\sqrt{m_i}}{c_i} \left(e^{\frac{1}{2}\alpha_i \cdot D(\phi-\psi)} - e^{\frac{1}{2}\alpha_{\Pi(i)} \cdot D(\phi-\psi)} \right) . \tag{3.23}$$

Clearly, most of the inner products on the left hand side of (3.23) must be zero. An economical set of inner products which (if possible) will allow (3.23) to be solved are:

$$\alpha_j \cdot D\alpha_i = \begin{cases} \alpha_i^2 & j = i \, ; \\ -\alpha_{\Pi(i)}^2 & j = \Pi(i) \, ; \\ 0 & \text{otherwise.} \end{cases} . \tag{3.24}$$

The latter are achievable provided it is possible to write

$$D\alpha_i = 2\left(\lambda_i - \lambda_{\Pi(i)}\right) , \tag{3.25}$$

where the vectors $\lambda_i,\ i = 1, \ldots, r$ are the fundamental weights satisfying

$$\alpha_i \cdot \lambda_j = \frac{\alpha_i^2}{2} \delta_{ij} , \qquad i, j = 1, \ldots, r , \tag{3.26}$$

and $\lambda_0 \equiv 0$. Note, the necessity to include the term labelled 0, creates certain problems. Without it, the Toda theory would be conformal and eqs(3.25) and (3.26) would suffice.

It is worth noting, since the permutation Π preserves the affine root diagram, that the coefficients n_i in the expression for α_0 in terms of the simple roots satisfy

$$n_i = n_{\Pi(i)} \,.$$

Using this fact, it is easy to check that the suggested relation (3.25) is at the very least consistent since

$$\sum_{i=0}^{r} n_i D\alpha_i = 2\sum_{i=0}^{r} n_i \left(\lambda_i - \lambda_{\Pi(i)}\right) = 2\sum_{i=0}^{r} n_i\lambda_i - 2\sum_{i=0}^{r} n_{\Pi(i)}\lambda_{\Pi(i)} \equiv 0\,.$$

As an example, take the case of $a_r^{(1)}$, whose roots are all of equal length, conventionally chosen to be $\alpha_i^2 = 2$. Let

$$\alpha_{\Pi(i)} = \alpha_{i+1}\,, \qquad i = 0, \ldots, r-1, \qquad \alpha_{\Pi(r)} = \alpha_0\,,$$

in which case Π is the generator of the elementary cyclic permutation symmetry of the $a_r^{(1)}$ Kač-Dynkin diagram. Then,

$$D\alpha_i = 2\left(\lambda_i - \lambda_{i+1}\right), \qquad D\alpha_r = 2\lambda_r\,, \qquad D\alpha_0 = -2\lambda_1\,. \tag{3.27}$$

Indeed, for this particular symmetry, D may be written explicitly

$$D = \sum_{i=1}^{r} \frac{4}{\alpha_i^2}\left(\lambda_i - \lambda_{i+1}\right)\lambda_i^T \equiv 2\sum_{i=1}^{r}\left(\lambda_i - \lambda_{i+1}\right)\lambda_i^T\,. \tag{3.28}$$

This is consistent with (3.11) since a direct computation indicates

$$\left(D + D^T\right)\alpha_k = 2\left(2\lambda_k - \lambda_{k+1} - \lambda_{k-1}\right) \equiv 2\alpha_k\,.$$

Since the simple roots are linearly independent, the required property $D + D^T = 2$ follows. Another way to see this is to note simply

$$\left(D + D^T\right) = 2\sum_{i=1}^{r}\left(2\lambda_i\lambda_i^T - \lambda_{i+1}\lambda_i^T - \lambda_i\lambda_{i+1}^T\right) = 2\sum_{i,j=1}^{r}\lambda_i C_{ij}\lambda_j^T = 2\,, \tag{3.29}$$

where C_{ij} is the a_r Cartan matrix. In passing, it is also noteworthy that the antisymmetric matrix $1 - D$ is given by the expression

$$1 - D = \sum_{i=1}^{r}\left(\lambda_{i+1}\lambda_i^T - \lambda_i\lambda_{i+1}^T\right), \tag{3.30}$$

which will be required later.

It is straightforward to satisfy (3.23) by taking

$$c_i = c_{\Pi(i)} = c_{i+1} \equiv c\,, \qquad s_i = s = \frac{2\sqrt{m_i}}{c_i\,\alpha_i^2} \equiv \frac{1}{c}\,, \tag{3.31}$$

leaving a single free parameter c. In this case, the roots form a single orbit under Π, but that may not always be so. (At this stage, it appears free parameters might be associated with orbits under Π.) Using (3.31) and (3.14), the $a_r^{(1)}$ defect potential associated with the cyclic permutation Π is

$$\mathcal{B} = \sum_{i=0}^{r}\left(d\,e^{\frac{1}{2}\alpha_i\cdot(D^T\phi+D\psi)} + \frac{1}{d}\,e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)}\right). \tag{3.32}$$

The story is not quite finished since there remain other pieces to be checked. The remaining terms in (3.15) at order $1/\lambda$ need to be examined to determine k_2. Since the exponential factors are expected generally to be different the terms that need to match are

$$\begin{aligned} -2\sqrt{m_i}\,[k_2, E_{\alpha_i}] &= d_i\left(\alpha_i\cdot Dk_1\mathbf{H} + \alpha_i\cdot D^T\mathbf{H}k_1\right) \\ &= d_i\left(\alpha_i\cdot(D+D^T)\mathbf{H}k_1 + \alpha_i\cdot D\,[k_1,\mathbf{H}]\right) \\ &= d_i\left(2\alpha_i\cdot\mathbf{H}k_1 + \alpha_i\cdot D\sum_{j=0}^{r}c_j\alpha_j E_{-\alpha_j}\right) \\ &= d_i\left(2\alpha_i\cdot\mathbf{H}k_1 + c_i\alpha_i^2E_{-\alpha_i} - c_{\Pi^{-1}(i)}\alpha^2_{\Pi^{-1}(i)}E_{-\alpha_{\Pi^{-1}(i)}}\right), \end{aligned} \tag{3.33}$$

where the expression (3.14) for k_1 has been used along with the facts (3.24). For the choice of roots corresponding to $a_r^{(1)}$ with the permutation and the conventions used above, eq (3.15) simplifies to

$$-[k_2, E_{\alpha_i}] = d^2\left(\alpha_i\cdot\mathbf{H}\sum_{j=0}^{r}E_{-\alpha_j} + E_{-\alpha_i} - E_{-\alpha_{i-1}}\right). \tag{3.34}$$

The natural algebra grading dictates that k_2 has the form

$$k_2 = \sum_{k,l=0}^{r} c_{kl}E_{-\alpha_k}E_{-\alpha_l}\,, \tag{3.35}$$

and therefore

$$-[k_2, E_{\alpha_i}] = \alpha_i\cdot\mathbf{H}\sum_{k=0}^{r}(c_{ki}+c_{ik})E_{-\alpha_k} + \sum_{k=0}^{r}c_{ki}\,(\alpha_i\cdot\alpha_k)\,E_{-\alpha_k}\,. \tag{3.36}$$

Hence, comparing with (3.34) yields

$$c_{ki}+c_{ik} = d^2\,, \qquad c_{ii} = \frac{d^2}{2}\,, \qquad c_{i-1\,i} = d^2\,, \qquad c_{i+1\,i} = 0\,, \qquad i,k = 0,\ldots,r \tag{3.37}$$

which allows for some considerable freedom in k_2.

On the other hand, $k_2 \equiv 0$ when evaluated in an $r+1$ dimensional representation for which

$$(E_{-\alpha_i})_{ab} = \delta_{ai}\delta_{b\,i-1}\,, \qquad a,b = 1,\dots,r+1\,. \tag{3.38}$$

Indeed, in that particular representation it can be checked directly that a complete expression for $\tilde{\mathcal{K}}$ is given by

$$\tilde{\mathcal{K}} = 1 + \frac{d}{\lambda}\sum_{i=0}^{r} E_{-\alpha_i}\,. \tag{3.39}$$

In general, at the next order, $1/\lambda^2$, there may be further constraints on k_2. To examine these it is necessary to analyse the following

$$\begin{aligned} k_2\mathbf{H}\cdot\nabla_\psi\mathcal{B} + \mathbf{H}k_2\cdot\nabla_\phi\mathcal{B} = -\sum_{i=0}^{r}\sqrt{m_i}\Big([k_3, E_{\alpha_i}]\, e^{\frac{1}{2}\alpha_i\cdot(D^T\phi + D\psi)} + \\ + \Big(E_{-\alpha_i}k_1 - k_1 E_{-\alpha_{\Pi^{-1}(i)}}\Big)\, e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)}\Big)\,, \end{aligned} \tag{3.40}$$

where, in the last term on the right hand side, the relation (3.19) has been used together with a reordering of the sum. Apart from an equation for k_3 there is a further equation involving k_2 by itself, which might, in principle, require further restrictions. Thus, inserting the expression for $\mathcal{B}$,

$$\frac{1}{2d}[k_2, \mathbf{H}]\cdot D^T\alpha_i = \sqrt{m_i}\,\Big(E_{-\alpha_i}k_1 - k_1 E_{-\alpha_{\Pi^{-1}(i)}}\Big)\,,$$

which can be shown to be identically satisfied using the already obtained expressions for k_1 and k_2, and using information concerning the Lie algebra of A_r. Specifically, it is important to note that the commutator $[E_{-\alpha_k}, E_{-\alpha_l}]$ vanishes except when $k = l \pm 1$. So, perhaps surprisingly, there is actually no additional constraint on k_2.

On the other hand, the equation for k_3 leads to

$$k_3 = \sum_{k,l,m=0}^{r} c_{klm}E_{-\alpha_k}E_{-\alpha_l}E_{-\alpha_m}\,,$$

which is of the expected form with respect to the grading alluded to above, with a set of relations among the coefficients. These are quite complicated and summarised below

$$\begin{aligned}
c_{ijk} + c_{ikj} + c_{jik} + c_{jki} + c_{kij} + c_{kji} &= d^3\,, \qquad i,j,k = 0,\dots,r\,;\\
c_{ijk} + c_{jik} + c_{jki} &= dc_{jk}\,, \qquad i,j = 0,\dots,r,\ k = j\pm 1\,;\\
c_{k\,i-1\,i} + c_{i-1\,k\,i} + c_{i-1\,i\,k} &= d^3\,, \qquad k \neq i+1, i, i-1, i-2\,;\\
c_{i+1\,k\,i} + c_{i+1\,i\,k} + c_{k\,i+1\,i} &= 0\,, \qquad k \neq i+2, i+1, i, i-1\,;\\
c_{i-1\,i\,i-1} + 2c_{i-1\,i-1\,i} &= d^3\,, \qquad c_{i+1\,i\,i+1} + 2c_{i+1\,i+1\,i} = 0\,;\\
c_{i\,i-1\,i-1} - c_{i-1\,i-1\,i} &= -\frac{d^3}{2} = c_{i+1\,i+1\,i} - c_{i\,i+1\,i+1}\,, \qquad i = 0,\dots,r\,;\\
c_{i-2\,i-1\,i} &= d^3 = 2c_{i-2\,i-1\,i} + c_{i-2\,i\,i-1}\,;\\
2c_{i+2\,i-1\,i} + 2c_{i-2\,i+1\,i} + c_{i+1\,i\,i-1} + c_{i-1\,i\,i+1} &= d^3\,;\\
c_{i+2\,i+1\,i} &= 0 = c_{i-1\,i-2\,i} + c_{i-1\,i\,i-2}\,, \qquad i = 0,\dots,r\,.
\end{aligned} \tag{3.41}$$

These relations refer to the general case a_r. However, for the special case of a_2, because of the identification of the labelling modulo $r+1$, some of the relations are modified and do not fit the general type. Also, these relations do not specify the coefficients of k_3 uniquely; there is further freedom.

We have no reason to suppose there is any problem, at least in principle, in continuing this iterative process to determine $\tilde{\mathcal{K}}$, but it does not lead easily to a closed form, and such an expression has not been found by alternative means. It remains an open question.

4. Momentum revisited

In section 2 it was pointed out that a defect can allow momentum to be conserved provided the momentum density is suitable modified. There, the example used was a pair of free scalar fields with a single defect at $x = 0$. In this section the possibility of defining a conserved momentum within the more general situation will be explored. The starting point is eq(2.3) in which each scalar field is replaced by a multi-component field and the products interpreted accordingly. Thus, using the defect conditions (3.2)

$$\begin{aligned}\frac{dP}{dt} &= \frac{1}{2}\left[(\partial_t\phi)^2 + (E\partial_t\phi + D\partial_t\psi - \nabla_\phi\mathcal{B})^2 - 2V(\phi)\right] - \\ &\quad -\frac{1}{2}\left[(\partial_t\psi)^2 + \left(D^T\partial_t\phi - F\partial_t\psi + \nabla_\psi\mathcal{B}\right)^2 - 2W(\psi)\right],\end{aligned} \tag{4.1}$$

where all fields on the right hand side are evaluated at $x = 0$. In order that (4.1) be the total time derivative of a functional of the fields ψ and ϕ evaluated at $x = 0$, the following relations must hold

$$\begin{aligned}\partial_t\phi\cdot(1 + E^TE - DD^T)\partial_t\phi &= 0\\ \partial_t\psi\cdot(1 - D^TD - F^TF)\partial_t\psi &= 0\\ \partial_t\phi\cdot(E^TD + DF)\partial_t\psi &= 0\\ \frac{1}{2}(\nabla_\phi\mathcal{B})^2 - \frac{1}{2}(\nabla_\psi\mathcal{B})^2 &= V(\phi) - W(\psi)\,.\end{aligned} \tag{4.2}$$

The first three of the required relations (4.2) follow directly from the already derived relationships between E, F and D, (3.11); the fourth requires a more careful examination but also follows from properties of the matrix D.

Assembling the pieces of the defect potential contained in eqs(3.14), (3.22),

$$\mathcal{B} = \sum_{i=0}^{r}\left[d_i e^{\frac{1}{2}\alpha_i\cdot(D^T\phi + D\psi)} + s_i e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)}\right] \equiv \sum_{i=0}^{r}(f_i + g_i)\,, \tag{4.3}$$

and evaluating the left hand side of the fourth of eqs(4.2) gives

$$\frac{1}{2}(\nabla_\phi\mathcal{B})^2 - \frac{1}{2}(\nabla_\psi\mathcal{B})^2 = \frac{1}{2}\sum_{i,j=0}^{r} d_i s_j\,\alpha_i\cdot D\alpha_j\, e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi) + \frac{1}{2}\alpha_j\cdot(D^T\phi + D\psi)}\,. \tag{4.4}$$

Using the basic properties of D expressed in eqs(3.24), (3.19), the right hand side of (4.4) may be rewritten

$$\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{r} s_j d_j \alpha_j^2\, e^{\frac{1}{2}\alpha_j\cdot(D+D^T)\phi} - \frac{1}{2}\sum_{j=0}^{r} s_{\Pi(j)} d_j \alpha_{\Pi(j)}^2\, e^{\frac{1}{2}\alpha_{\Pi(j)}\cdot D(\phi-\psi)+\frac{1}{2}\alpha_j\cdot(D^T\phi+D\psi)} = \\
&\quad = \frac{1}{2}\sum_{j=0}^{r} s_j d_j \alpha_j^2\, e^{\alpha_j\cdot\phi} - \frac{1}{2}\sum_{j=0}^{r} s_{\Pi(j)} d_j \alpha_{\Pi(j)}^2\, e^{\alpha_j\cdot\psi} \\
&\quad = V(\phi) - V(\psi)
\end{aligned}$$

where V is the affine Toda potential, provided, for each $j = 0, \dots, r$,

$$\frac{1}{2} s_j d_j \alpha_j^2 = n_j = \frac{1}{2} s_{\Pi(j)} d_j \alpha_{\Pi(j)}^2 \,.$$

Since Π is a symmetry of the extended root diagram, $\alpha_j^2 = \alpha_{\Pi(j)}^2$, and $s_j = s_{\Pi(j)} = n_j/d_j$.

Finally, similar manipulations lead to the conclusion

$$\frac{dP}{dt} = \frac{dU}{dt}\,, \qquad U = -\sum_{i=0}^{r} d_i e^{\frac{1}{2}\alpha_i\cdot(D^T\phi+D\psi)} + \sum_{i=0}^{r} s_i e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)}\Bigg|_{x=0} \,. \tag{4.5}$$

Hence, the generalised conserved momentum may be written

$$\mathcal{P} = P - U = P + \sum_{i=0}^{r}(f_i - g_i)\,. \tag{4.6}$$

This is to be compared with the standard expression for the conserved energy

$$\mathcal{E} = E + \mathcal{B} = E + \sum_{i=0}^{r}(f_i + g_i)\,, \tag{4.7}$$

and it is worth noting that the 'light-cone' versions of these

$$\mathcal{P}_{\pm} = \frac{1}{2}(\mathcal{E} \pm \mathcal{P}) \tag{4.8}$$

involve only the f_i or g_i, respectively, meaning (because of the relationship between d_i and s_i) that a Lorentz transformation of the bulk quantities can be compensated by a change of scale in d_i, just as was found in the linear case (2.10). If there are several defects located at different places the total energy and the total modified momentum will be conserved with contributions arising at each defect.

5. Higher spin conserved charges

Given the existence of a Lax pair incorporating the defect it is certainly expected that there will be an infinite number of conserved charges whose generating function will be of the form [11],

$$\mathcal{Q}(\lambda) = \mathrm{tr}\left(\exp\left[\int_{-\infty}^{a} dx\, \widehat{a}_x^{<}(\lambda)\right]\mathcal{K}(\lambda)\exp\left[\int_{b}^{\infty} dx\, \widehat{a}_x^{>}(\lambda)\right]\right). \tag{5.1}$$

The quantity $\mathcal{Q}(\lambda)$ is time-independent because of the zero curvature condition satisfied by the two gauge fields $\widehat{a}_x^<$ and $\widehat{a}_x^>$, and because of the condition on $\mathcal{K}$ expressed by (3.7). Expanding $\mathcal{Q}(\lambda)$ as a Laurent series in λ reveals the conserved quantities as the coefficients of the various powers of λ that occur in the expansion. On the other hand, it has not yet proved possible to calculate the conserved charges from the Lax pair despite it being feasible in principle. In fact, there is a subtlety since not all charges that are conserved in the bulk survive in the presence of a defect - indeed that might be expected given the experience gained by dealing with boundaries. There, typically all 'momentum-like' charges are lost although 'energy-like' charges are preserved, albeit modified. In the presence of a defect it was demonstrated in the previous section that momentum and energy themselves (spin $s = \pm 1$) are both preserved, at least after suitable adjustments have been made to include a contribution to the momentum from the defect. A glance at the expression for $\tilde{\mathcal{K}}$ provided by (3.39) suggests there is a lack of balance between λ and $1/\lambda$ (or equivalently d and $1/d$) which might indicate that charges of positive spins should behave generally quite differently to those of negative spin. To illustrate this, it is enough to examine the charges which would have spin $s = \pm 2$ in the bulk.

A general ansatz for a spin $s = \pm 3$ bulk density $T_{\pm 3}$ (using light-cone coordinates $x^\pm = (t \pm x)/\sqrt{2}$) reads (see [11])

$$T_{\pm 3} = \frac{1}{3} A_{abc} \partial_\pm \phi_a \partial_\pm \phi_b \partial_\pm \phi_c + B_{ab} \partial_\pm^2 \phi_a \partial_\pm \phi_b \, , \tag{5.2}$$

where the coefficients A_{abc} are completely symmetric and the coefficients B_{ab} are antisymmetric. An explicit calculation reveals that the continuity equation

$$\partial_\mp T_{\pm 3} = \partial_\pm \Theta_{\pm 1}$$

is satisfied for the choice

$$\Theta_{\pm 1} = -\frac{1}{2} B_{ab} \partial_\pm \phi_a V_b, \tag{5.3}$$

provided

$$A_{abc} V_a + B_{ab} V_{ac} + B_{ac} V_{ab} = 0 \, , \tag{5.4}$$

where

$$V_b = \frac{\partial V}{\partial \phi_b} \, , \qquad V_{bc} = \frac{\partial^2 V}{\partial \phi_b \partial \phi_c} \, .$$

As explained in [11] the latter set of equations determine A_{abc} and B_{ab} up to an overall multiplicative constant. It is often convenient to write (5.4) in a field independent way, in which case for each simple root α_i (or α_0)

$$A_{abc} (\alpha_i)_c + B_{cb} (\alpha_i)_c (\alpha_i)_a + B_{ca} (\alpha_i)_c (\alpha_i)_b = 0 \, . \tag{5.5}$$

Then, on careful inspection (5.5) requires B_{ab} to be a constant multiple of the right hand side of (3.30) and, therefore, without losing generality, it is convenient to write

$$B = 1 - D \, . \tag{5.6}$$

Light-cone coordinates are not convenient for models with a defect but a suitable rearrangement of (5.2) is

$$\partial_t(T_{\pm 3} - \Theta_{\pm 1}) = \pm\partial_x(T_{\pm 3} + \Theta_{\pm 1})\,, \tag{5.7}$$

leading to candidate bulk contributions to spin 2 conserved charges of the form

$$Q_{\pm 2} = \int_{-\infty}^{0} dx\,(T^{\phi}_{\pm 3} - \Theta^{\phi}_{\pm 1}) + \int_{0}^{\infty} dx\,(T^{\psi}_{\pm 3} - \Theta^{\psi}_{\pm 1})\,.$$

Then, the time derivatives of the candidate charges will be given by terms evaluated at the position of the defect

$$\frac{dQ_{\pm 2}}{dt} = \pm\left[T^{\phi}_{\pm 3} + \Theta^{\phi}_{\pm 1}\right]_0 \mp \left[T^{\psi}_{\pm 3} + \Theta^{\psi}_{\pm 1}\right]_0\,. \tag{5.8}$$

Provided the right hand side of (5.8) may be written as the time derivative of a functional, $\mathcal{B}_{\pm 2}$, depending upon the two fields ϕ and ψ evaluated at $x = 0$, the quantities

$$\widehat{Q}_{\pm 2} = Q_{\pm 2} - \mathcal{B}_{\pm 2}(\phi(0,t), \psi(0,t))$$

will be conserved.

To facilitate implementing the defect conditions in (5.8) it is useful to reorganise (3.2) to read as follows

$$\begin{aligned}
\sqrt{2}\partial_-\phi &= D\partial_t(\phi - \psi) + \nabla_\phi \mathcal{B}, \\
\sqrt{2}\partial_-\psi &= D^T\partial_t(\psi - \phi) - \nabla_\psi \mathcal{B},
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
\sqrt{2}\partial_+\phi &= D^T\partial_t\phi + D\partial_t\psi - \nabla_\phi \mathcal{B}\,, \\
\sqrt{2}\partial_+\psi &= D\partial_t\psi + D^T\partial_t\phi + \nabla_\psi \mathcal{B}\,.
\end{aligned} \tag{5.10}$$

Notice that the second pair also implies

$$\sqrt{2}\partial_+(\phi - \psi) = -(\nabla_\phi\mathcal{B} + \nabla_\psi\mathcal{B})$$

independently of the matrix D.

The two different groupings are equivalent but convenient since the boundary conditions will be used to replace light-cone derivatives evaluated at the defect. It is quite complicated to implement these conditions in (5.8). However, before examining the full detail it is soon apparent there is a distinction between the two spins. To see this, examine the highest order terms in the time derivatives of the fields appearing on the right hand side of (5.8) after the defect conditions have been implemented. For example, in the case of spin -2 a typical one of these up to a numerical factor is

$$\partial_t\phi_p\partial_t\phi_q\partial_t\phi_r A_{abc}(D_{ap}D_{bq}D_{cr} + D_{pa}D_{qb}D_{rc})\,. \tag{5.11}$$

Clearly, because of (3.20) the expression (5.11) does not vanish and it cannot be written as a total time derivative. Hence the candidate charge Q_{-2} cannot be modified in any simple way to yield a conserved quantity. On the other hand, this particular type of term vanishes identically for the spin +2 case. Thus, at best Q_2 may have a conserved modification, and the next part of this section will explore that possibility in detail. The key lies in the fact that the same combination of derivatives of fields ϕ and ψ appears on the right hand sides of the 'plus' parts of both of the boundary conditions supplied by (5.10).

To facilitate checking it is useful to make a small change of notation, abbreviating the combination $D^T\phi + D\psi$ to R and $(\nabla_\phi \mathcal{B})_a$ to $\mathcal{B}^\phi_a$, $(\nabla_\phi V)_a$ to V^ϕ_a, etc. Then, the terms which need to be examined are the following

$$\begin{aligned}&\frac{1}{6\sqrt{2}}A_{abc}\Big[(\partial_t R_a-\mathcal{B}^\phi_a)(\partial_t R_b-\mathcal{B}^\phi_b)(\partial_t R_c-\mathcal{B}^\phi_c)-(\partial_t R_a-\mathcal{B}^\psi_a)(\partial_t R_b-\mathcal{B}^\psi_b)(\partial_t R_c-\mathcal{B}^\psi_c)\Big]+\\&+\frac{1}{2\sqrt{2}}B_{ab}\Big[(2\partial_t^2R_a-2\partial_t\mathcal{B}^\phi_a+V^\phi_a)(\partial_t R_b-\mathcal{B}^\phi_b)-(2\partial_t^2R_a+2\partial_t\mathcal{B}^\psi_a+V^\psi_a)(\partial_t R_b+\mathcal{B}^\psi_b)\Big]-\\&-\frac{1}{2\sqrt{2}}B_{ab}\Big[(\partial_t R_a-\mathcal{B}^\phi_a)V^\phi_b-(\partial_t R_a+\mathcal{B}^\psi_a)V^\psi_b\Big]\,.\end{aligned} \tag{5.12}$$

Clearly, as already noted, all the terms with three time derivatives cancel.

The terms with two time derivatives, after making use of the explicit form of the defect potential and the basic property (5.5), are seen to be a total time derivative

$$-\frac{1}{\sqrt{2}}\partial_t\,\Big[B_{ab}\partial_t R_a(\mathcal{B}^\phi_b+\mathcal{B}^\psi_b)\Big]\,. \tag{5.13}$$

The terms with a single time derivative gather together to

$$\frac{1}{2\sqrt{2}}\,A_{abc}\partial_t R_a(\mathcal{B}^\phi_b-\mathcal{B}^\psi_b)(\mathcal{B}^\phi_c+\mathcal{B}^\psi_c)+\frac{1}{\sqrt{2}}\,B_{ab}(\partial_t\mathcal{B}^\phi_a\mathcal{B}^\phi_b-\partial_t\mathcal{B}^\psi_a\mathcal{B}^\psi_b+(V^\phi_a-V^\psi_a)\partial_t R_b)\,,$$

which is also a total derivative

$$\partial_t\left(\frac{1}{4\sqrt{2}}\sum_{i,j}\alpha_i\cdot D(1-D)\alpha_j\,e^{\frac{1}{2}\alpha_i\cdot D(\phi-\psi)}e^{\frac{1}{2}\alpha_j\cdot R}\right). \tag{5.14}$$

This follows, on using (5.5), (5.6) and the basic defining property of D, (3.19).

The terms without any time derivatives are

$$-\frac{1}{6\sqrt{2}}\,A_{abc}(\mathcal{B}^\phi_a\mathcal{B}^\phi_b\mathcal{B}^\phi_c-\mathcal{B}^\psi_a\mathcal{B}^\psi_b\mathcal{B}^\psi_c)-\frac{1}{\sqrt{2}}\,B_{ab}(V^\phi_a\mathcal{B}^\phi_b+V^\psi_a\mathcal{B}^\psi_b)\,; \tag{5.15}$$

using (5.5), the explicit expression for the defect potential, and the defining property of D these are shown to exactly cancel. Hence, a suitably modified spin 2 charge is conserved.

For the example we have been using, the spin +2 charge is preserved, the spin -2 is lost. On the other hand, a moment's thought reveals that a slightly different example would interchange the rôles of these two charges. Return to the equation defining $\tilde{\mathcal{K}}$, (3.13). To solve this it was supposed $\tilde{\mathcal{K}}$ had an expansion in inverse powers of the spectral parameter λ. If instead $\tilde{\mathcal{K}}$ was presumed to have an expansion in positive powers of λ, the properties of D, the relationships between D, E and F and the boundary potential would be slightly different but another consistent solution would be obtained. That is,

$$E = F = 1 + D^T\,, \qquad D + D^T = -2\,, \tag{5.16}$$

with

$$\begin{aligned}\sqrt{2}\partial_-\phi &= -D^T\partial_t\phi - D\partial_t\psi + \nabla_\phi\mathcal{B}\,,\\ \sqrt{2}\partial_-\psi &= -D^T\partial_t\phi - D\partial_t\psi - \nabla_\psi\mathcal{B}\,,\end{aligned} \tag{5.17}$$

$$\begin{aligned}\sqrt{2}\partial_+\phi &= -D\partial_t(\phi-\psi) - \nabla_\phi\mathcal{B}\,,\\ \sqrt{2}\partial_+\psi &= -D^T\partial_t(\psi-\phi) + \nabla_\psi\mathcal{B}\,,\end{aligned} \tag{5.18}$$

and D is replaced by $-D$ in the defect potential. For this solution, the spin -2 charge is conserved instead. This means that the presence of two defects of opposing types located at different positions would entail a complete loss of the spin two charges. Since even spin charges are able to distinguish between particles and antiparticles in the $a_r^{(1)}$ affine Toda field theory, the distinction between particles and antiparticles should presumably be lost in a typical situation with many defects. A full discussion of conserved charges would require deducing the form they take from (5.1) and that has not yet been done. It is conceivable the apparently broken charges are repaired by assembling combinations with several different spins, but that remains to be seen. Such a possibility is certainly feasible since the defect explicitly breaks Lorentz invariance. In the presence of a boundary, something similar needed to be done since the 'energy-like' combinations of positive and negative spins have to be used to assemble conserved quantities.

6. Defect conditions from Bäcklund transformations

The argument of section 3 aims to constrain the boundary lagrangian by imposing the condition that the resulting boundary conditions lead to the existence of a gauge transformation between the two zero-curvature Lax pairs on each side of the defect. In this case, one is guaranteed the classical integrability of the model. In the case of sine-Gordon theory, the resulting boundary conditions are exactly the same as the auto-Bäcklund transformation for sine-Gordon theory. In some ways this is a remarkable result, since the Bäcklund transformations are equations which are true for all x, the corresponding boundary conditions could be imposed at any value of x without changing the value of the fields far away from the boundary. Indeed the form of the boundary potential determines the boundary conditions, which assumed true for all x can be cross-differentiated to determine the bulk equations of motion.

The purpose of this section is to work from the opposite point of view, and ask when boundary conditions of the form (3.2), which arise from a boundary lagrangian coupling two sets of n scalar fields ψ, ϕ can be interpreted as a Bäcklund transformation leading to Lorentz invariant second order equations for the fields in the bulk. Under some fairly mild assumptions the solutions for D, E and F found in section 3 will be recovered.

For these purposes it will be convenient to think of ϕ and ψ as components of a $2n$ dimensional vector $\Psi = (\phi, \psi)$. In this notation the boundary condition can be written

$$\partial_x \Psi = M \partial_t \Psi + \begin{pmatrix} -\mathcal{B}_\phi \\ \mathcal{B}_\psi \end{pmatrix}, \tag{6.1}$$

where

$$M = \begin{pmatrix} E & D \\ D^T & -F \end{pmatrix}. \tag{6.2}$$

Differentiating with respect to x and t and removing the cross-derivatives reveals

$$\partial_x^2 \Psi = M^2 \partial_t^2 \Psi + (MN + NM)\, \partial_t \Psi + N \begin{pmatrix} -\mathcal{B}_\phi \\ \mathcal{B}_\psi \end{pmatrix}, \tag{6.3}$$

where

$$N = \begin{pmatrix} -\mathcal{B}_{\phi\phi} & -\mathcal{B}_{\phi\psi} \\ \mathcal{B}_{\psi\phi} & \mathcal{B}_{\psi\psi} \end{pmatrix}. \tag{6.4}$$

Demanding the resulting equations for ϕ and ψ be decoupled, and Lorentz-invariant, leads immediately to

$$M^2 = 1 \tag{6.5}$$

$$MN + NM = 0 \tag{6.6}$$

$$\mathcal{B}_\phi^2 - \mathcal{B}_\psi^2 = 2(V(\phi) - W(\psi)), \tag{6.7}$$

where V and W should be the bulk potentials. The next step is to solve these three equations in turn. In components, (6.5) is equivalent to

$$E^2 + DD^T = 1, \qquad F^2 + D^T D = 1, \qquad ED - DF = 0, \qquad D^T E - FD^T = 0. \tag{6.8}$$

For a real boundary potential, E and F may be taken to be real antisymmetric matrices. So, the first of these equations may be rewritten

$$DD^T = 1 - E^2 = (1 - E)(1 - E)^T, \tag{6.9}$$

while the last of equations (6.8) are the same. But, if E is real and antisymmetric, it has purely imaginary eigenvalues and hence $(1 - E)$ is invertible. Then

$$(1 - E)^{-1} D((1 - E)^{-1} D)^T = 1, \tag{6.10}$$

indicating that $D = (1 - E)R$ where $R \in O(n)$. By making an orthogonal transformation on, say, the basis of ψ, R may be replaced by the unit matrix, and the relationship $D = 1 - E$ is recovered. It follows D is also invertible and from this it is easy to prove $F = E = 1 - D$.

Eq(6.6) is a second order linear differential equation for $\mathcal{B}$, which simplifies provided a different basis corresponding to linear combinations of ϕ and ψ is used. Choosing variables

$$\mu = D^T \phi + D\psi$$

$$\nu = -D(\phi - \psi), \tag{6.11}$$

and making the change to this basis, noting

$$\begin{pmatrix} \partial_\phi \\ \partial_\psi \end{pmatrix} = \begin{pmatrix} D & -D^T \\ D^T & D^T \end{pmatrix} \begin{pmatrix} \partial_\mu \\ \partial_\nu \end{pmatrix}, \tag{6.12}$$

the condition (6.6) becomes

$$2 \begin{pmatrix} -D & -D^T \\ -D^T & D^T \end{pmatrix} \begin{pmatrix} 0 & \mathcal{B}_{\mu\nu} \\ \mathcal{B}_{\nu\mu} & 0 \end{pmatrix} \begin{pmatrix} D^T & D \\ -D & D \end{pmatrix} = 0\,, \tag{6.13}$$

which is solved if $\mathcal{B}_{\mu\nu} = 0$, with solution $\mathcal{B} = \mathcal{B}_1(\mu)+\mathcal{B}_2(\nu)$.

Finally, (6.7) is also relatively neat in terms of the variables μ and ν and can be written

$$\mathcal{B}_\phi^2 - \mathcal{B}_\psi^2 = -4\, D^T_{\alpha\beta}\, \frac{\partial \mathcal{B}_1}{\partial \mu_\alpha} \frac{\partial \mathcal{B}_2}{\partial \nu_\beta} = 2(V(\phi) - W(\psi))\,. \tag{6.14}$$

Bearing in mind the expected forms of V and W, it is convenient to take,

$$\mathcal{B}_1 = \sum_i c_i e^{v^{(i)}\cdot\mu}, \quad \mathcal{B}_2 = \sum_j d_j e^{w^{(j)}\cdot\nu}\,. \tag{6.15}$$

Since $\phi = \frac{1}{2}(\mu-\nu)$, a non-zero $V(\phi)$ requires $v^{(i)} = -w^{(j)}$ for some i, j. By a suitable choice of ordering, it is convenient to arrange that $v^{(i)} = -w^{(i)}$. Now, if the vector space spanned by ϕ is denoted by S_ϕ, and the vector space spanned by ψ is denoted by S_ψ, then (6.7) implies that for each i and j one of the following three possibilities must hold

$$\begin{aligned} v^{(i)}\cdot\mu - v^{(j)}\cdot\nu &\in S_\phi \\ v^{(i)}\cdot\mu - v^{(j)}\cdot\nu &\in S_\psi \\ D^T_{\alpha\beta} v^{(i)}_\alpha v^{(j)}_\beta &= 0\,. \end{aligned} \tag{6.16}$$

The first condition holds if $i = j$, and indeed it can be shown that it holds only in this case. To see this, construct a contradiction. Suppose $v^{(i)}\cdot\mu - v^{(j)}\cdot\nu \in S_\phi$, then

$$v^{(i)}\cdot\mu - v^{(j)}\cdot\nu - (v^{(j)}\cdot\mu - v^{(j)}\cdot\nu) = (v^{(i)} - v^{(j)})\cdot\mu \in S_\phi\,.$$

But, by the definition of μ,

$$(v^{(i)} - v^{(j)})\cdot(\mu - D^T\phi) = (v^{(i)} - v^{(j)})\cdot D\psi\,.$$

The left-hand side of this equation is in S_ϕ but the right hand side cannot be in S_ϕ, and indeed cannot vanish since $i \neq j$ and D is invertible.

By similar considerations, for fixed i the second of the conditions (6.16) can only be satisfied for at most one j which will not be equal to i. Again, choosing labels appropriately, this condition is taken to hold for $j = i+1$. If it is also supposed this process terminates and that for some N, $v^{(N+1)} = v^{(1)}$, then

$$\sum_{i=1}^{N} \left(v^{(i)}\cdot\mu - v^{(i)}\cdot\nu\right) = \sum_{i=1}^{N} \left(v^{(i)}\cdot\mu - v^{(i+1)}\cdot\nu\right). \tag{6.17}$$

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The left-hand side of (6.17) is clearly in S_ϕ whilst the right is in S_ψ and thus both sides must vanish. This in turn implies that

$$\sum_{i=1}^{N} v^{(i)} = 0. \tag{6.18}$$

Finally let us turn to the third of conditions (6.16). This condition should hold whenever the first two do not, that is, if $i \neq j$ and $i+1 \neq j$. If $|i-j| > 1$ then

$$\begin{aligned} D^T_{\alpha\beta} v^{(i)}_\alpha v^{(j)}_\beta &= 0 \\ D^T_{\alpha\beta} v^{(j)}_\alpha v^{(i)}_\beta &= D_{\alpha\beta} v^{(i)}_\alpha v^{(j)}_\beta = 0\,, \end{aligned} \tag{6.19}$$

implying that $v^{(i)} \cdot v^{(j)} = 0$ and $E_{\alpha\beta} v^{(i)}_\alpha v^{(j)}_\beta = 0$. Next, consider the remaining possibility with $j = i-1$ for which the third of conditions (6.16) also holds. In this case,

$$\begin{aligned} 0 = D^T_{\alpha\beta} v^{(i)}_\alpha v^{(i-1)}_\beta &= -\sum_{k\neq i}^{N} D^T_{\alpha\beta} v^{(k)}_\alpha v^{(i-1)}_\beta \\ &= -D^T_{\alpha\beta} v^{(i-2)}_\alpha v^{(i-1)}_\beta - D^T_{\alpha\beta} v^{(i-1)}_\alpha v^{(i-1)}_\beta \\ &= -D^T_{\alpha\beta} v^{(i-2)}_\alpha v^{(i-1)}_\beta - v^{(i-1)} \cdot v^{(i-1)} \end{aligned} \tag{6.20}$$

and

$$0 = D^T_{\alpha\beta} v^{(i-1)}_\alpha v^{(i-2)}_\beta = D_{\alpha\beta} v^{(i-2)}_\alpha v^{(i-1)}_\beta\,. \tag{6.21}$$

From these (on resetting $i \to i+1$) one may deduce

$$2v^{(i-1)} \cdot v^{(i)} + v^{(i)} \cdot v^{(i)} = 0\,, \tag{6.22}$$

and hence

$$E_{\alpha\beta} v^{(i-1)}_\alpha v^{(i)}_\beta = v^{(i-1)} \cdot v^{(i)}\,. \tag{6.23}$$

At this stage, all the inner products of the vectors (up to an overall scale factor) have been deduced. Taking $N = r+1$, it is clear the inner products are those between simple roots of A_r, and thus it is possible to put $v^{(i)} \sim \alpha_i$. The action of the matrix E on any pair of vectors $v^{(i)}$, $v^{(j)}$, is known and, after a little algebra, it is found that in terms of the fundamental weights λ_i of A_r

$$E = \sum_{i=1}^{r} (\lambda_{i+1}\lambda_i^T - \lambda_i\lambda_{i+1}^T)\,, \tag{6.24}$$

where $\lambda_{r+1} = \lambda_N = 0$. Also, it is found that

$$D = 1 - E = 2\sum_{i=1}^{r} \lambda_i(\lambda_i^T - \lambda_{i+1}^T)\,, \tag{6.25}$$

as before. At this point, everything is now essentially known and the bulk affine Toda potentials will be recovered from (6.7).

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7. One soliton solution

To investigate what happens to a single soliton solution the defect conditions (3.2) for the affine $a_r^{(1)}$ Toda theories will be used in the context of complex valued fields. Using Hirota's formalism [20, 22] the single soliton solution can be written in the following manner

$$\phi_{(a)} = -\sum_{j=0}^{r} \alpha_j \ln \tau_j \,, \qquad \tau_j = 1 + E_a \omega^{aj} \,, \qquad \psi_{(a)} = -\sum_{j=0}^{r} \alpha_j \ln \sigma_j \,, \qquad \sigma_j = 1 + \Lambda E_a \omega^{aj} \,, \tag{7.1}$$

with

$$E_a = e^{a_a x + b_a t + \xi} \,, \qquad (a_a,\ b_a) = m_a(\cosh\theta, \sinh\theta) \,, \qquad m_a^2 = a_a^2 - b_a^2 = 4\sin^2\left(\frac{a\pi}{r+1}\right) ,$$

and $\omega = \exp(2\pi i/(r+1))$. It is presumed the rapidity θ and the type of soliton a is the same on both sides of the defect and the additional factor Λ represents the effect of the defect on the soliton as it passes through. Using the properties of D (3.19), and multiplying on the left by α_k, the following two equations which the single soliton solution has to satisfy are obtained,

$$\begin{aligned}
&2\frac{\tau_k'}{\tau_k} - \frac{\tau_{k+1}'}{\tau_{k+1}} - \frac{\tau_{k-1}'}{\tau_{k-1}} + \frac{\dot\tau_{k+1}}{\tau_{k+1}} - \frac{\dot\tau_{k-1}}{\tau_{k-1}} - 2\frac{\dot\sigma_k}{\sigma_k} + 2\frac{\dot\sigma_{k-1}}{\sigma_{k-1}} - \\
&-d\left(\frac{\tau_{k+1}}{\tau_k}\frac{\sigma_{k-1}}{\sigma_k} - \frac{\tau_k}{\tau_{k-1}}\frac{\sigma_{k-2}}{\sigma_{k-1}}\right) - \frac{1}{d}\left(\frac{\tau_{k-1}}{\tau_k}\frac{\sigma_k}{\sigma_{k-1}} - \frac{\tau_k}{\tau_{k+1}}\frac{\sigma_{k+1}}{\sigma_k}\right) = 0 \,, \\
&2\frac{\sigma_k'}{\sigma_k} - \frac{\sigma_{k+1}'}{\sigma_{k+1}} - \frac{\sigma_{k-1}'}{\sigma_{k-1}} + \frac{\dot\sigma_{k+1}}{\sigma_{k+1}} - \frac{\dot\sigma_{k-1}}{\sigma_{k-1}} - 2\frac{\dot\tau_k}{\tau_k} + 2\frac{\dot\tau_{k+1}}{\sigma_{k+1}} + \\
&+d\left(\frac{\tau_{k+1}}{\tau_k}\frac{\sigma_{k-1}}{\sigma_k} - \frac{\tau_{k+2}}{\tau_{k+1}}\frac{\sigma_k}{\sigma_{k+1}}\right) - \frac{1}{d}\left(\frac{\tau_{k-1}}{\tau_k}\frac{\sigma_k}{\sigma_{k-1}} - \frac{\tau_k}{\tau_{k+1}}\frac{\sigma_{k+1}}{\sigma_k}\right) = 0 \,.
\end{aligned}$$

Substituting the expressions for τ and σ reveals two equations in E of order five, which are compatible, and satisfied by the following expression for Λ

$$\Lambda = \left(\frac{ie^\theta + de^{i\zeta_a}}{ie^\theta + de^{-i\zeta_a}}\right), \qquad \zeta_a = \frac{\pi a}{r+1} \,. \tag{7.2}$$

Thus, the effect of the defect is to delay or advance the soliton as it passes. For the case of a self-conjugate soliton, that is for a soliton corresponding to $a = (r+1)/2$ (with r odd), the delay is real and identical with the expression already found for the sine-Gordon soliton [1]

$$\Lambda = \left(\frac{e^\theta + d}{e^\theta - d}\right).$$

On the other hand, for a conjugate pair of solitons, a and $\bar a = (r+1-a)$, the values of the delays are given by

$$\Lambda_{(a)} = \left(\frac{ie^\theta + de^{i\zeta_a}}{ie^\theta + de^{-i\zeta_a}}\right), \qquad \Lambda_{(\bar a)} = \left(\frac{ie^\theta - de^{-i\zeta_a}}{ie^\theta - de^{i\zeta_a}}\right) \equiv \bar\Lambda_a \,. \tag{7.3}$$

As already noted in [1], self-conjugate solitons can be absorbed or emitted for special values of real rapidity for which the delay or its inverse is infinite.

Taking the other boundary conditions, that is those that allow the spin -2 charge to be conserved instead, the corresponding expression for Λ is obtained by replacing θ with $-\theta$.

8. Summary and discussion

The purpose of this article has been to explore further the ideas introduced in [1] and analyse the extent to which the lagrangian description of a defect applies to affine Toda field theories other than the sine/sinh-Gordon model. Curiously, the $a_r^{(1)}$ models appear to be singled out, which is not quite what was anticipated since the affine Toda field theories using any of the affine root data have hitherto had rather similar properties with relatively small differences of detail. Perhaps a lack of sufficient imagination is the real problem and only time will tell. On the other hand, the description of a defect is very close to the Bäcklund transformation idea, and Bäcklund transformations were found first for the $a_r^{(1)}$ models [23], being rather more subtle for the other affine Toda field theories. After first noticing the association between a defect and a Bäcklund transformation it was expected its characteristically striking property of creating or annihilating solitons might show up in the guise of permitting a defect to change soliton number (for the use of Bäcklund transformations in this context for the $a_r^{(1)}$ case, see [24]). After all, a genuine defect will violate topological charge since there is no requirement for a field (or its space derivative) to be continuous through a defect. However, that interesting possibility does not appear to be realised.

Another intriguing possibility is the potential for controlling solitons by suitably tuning delays. It is known there are physical situations where the sine-Gordon solitons are relevant (a Josephson junction, for example) and it is interesting to speculate on what kinds of physically realisable mechanisms might produce a defect of the type described here. If such mechanisms exist then they could be used to control the arrival times of signals propagated by solitons. There is already literature concerning the effects of impurities on the motion of 'solitons' within various models (see, for example, [25]) although the emphasis is rather different from that of the approach adopted in this article.

It was not the purpose of this article to explore the associated quantum field theories. However, there are two interesting domains. The real field theories, whose particle spectrum is a set of scalar particles, distinguished by the eigenvalues of the higher spin charges, which are likely to have a set of transmission phase factors $T_a(\theta_a)$, one for each particle type, dependent upon the defect parameter and on the bulk coupling constant in addition to the rapidity of the particle. It is expected that as the bulk coupling tends to zero the transmission factor approaches its classical value dependent upon the defect parameter and rapidity. One interesting question is the fate of the weak-strong coupling duality under the transformation of the bulk coupling $\beta \to 4\pi/\beta$. On the other hand, the complex field theory, whose particle spectrum is conjectured to be a set of multiplets of sizes coinciding with the dimensions of the fundamental representations of A_r [22] (although not all of these have corresponding classical 'soliton' solutions [26]), is expected to have matrix transmission factors, which will also satisfy a Yang-Baxter type relation to ensure compatibility with the bulk S-matrix on either side of the defect, in addition to having a suitable classical limit. Determining these various quantum properties is a matter for further investigations beyond the scope of the present article.

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